Int. J. Solids Structures Vol. 25, No. 5, pp. 563–575, 1989
Printed in Great Britain.

0020–7683/89 \$3.00 + .00

NONLINEAR VIBRATIONS OF IN-PLANE LOADED, IMPERFECT, ORTHOTROPIC PLATES USING THE PERTURBATION TECHNIQUE

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(*Received* 16 *July* 1988)

Abstract—A perturbation technique is used to study the effects of in-plane inertia, rotary inertia, and shear deformation on the nonlinear free vibration response of an imperfect, in-plane loaded orthotropic plate. The von Karman type governing equilibrium equations of the plate correspond to those of a recently proposed shear deformation theory which employs parabolic shear strain variation across the thickness. The perturbation parameter is taken as the thickness to side length ratio of the plate. By expressing the generalized displacements in the form of a truncated power series of the perturbation parameter, the five governing equations of the problem under consideration are reduced to a single second order ordinary differential equation in terms of the transverse displacement. The solution of this equation is obtained by the method of multiple scales. Numerical results illustrate the influence of various parameters under consideration.

INTRODUCTION

A very large part of the theoretical investigation carried out on the vibration of plates subjected to in-plane loading is limited to linear small deformation theory as discussed by Bert (1982) and Leissa (1981). The inclusion of geometric nonlinearities while analysing the response of in-plane loaded plates is essential due to the presence of in-plane loading. Free vibration of plates with in-plane loading are of two types.

In the first type the free vibration characteristics of the plate in the presence of in-plane load is carried out. In this case the in-plane load may be constant or it may be periodically varying in time in which case the parametric vibrations are to be considered. A short history of this class of problem and the related references may be found in the papers by Pasic and Herrmann (1983, 1984).

The influence of in-plane loading, initial imperfections, in-plane inertia, and the geometric nonlinearities (taken individually) on the dynamic response of structures has been considered extensively by Bolotin (1964). For the first time, Pasic and Herrmann (1984) gave a general formulation for the free vibration analysis of plates treating all the parameters simultaneously. The important finding of Pasic and Herrmann (1984) was that the influence of in-plane inertia is considerable in nonslender plates when they are parametrically excited and may be neglected when the plate is subjected to a time independent in-plane load of constant magnitude.

Another more oftenly neglected aspect in the analysis of in-plane loaded plates is the influence of deformability of the loaded edges. By finding the exact solutions to the in-plane equilibrium equations, so as to satisfy the applied load exactly, Pasic and Herrmann (1983) investigated the effect of loaded edge deformability on the buckling and vibration of plates. They concluded that, for square (or nearly square) plates, the edge deformations are to be considered while they may be neglected for long rectangular plates.

In the second type of plate problems with in-plane load, the influence of initial buckle due to in-plane preload on the subsequent vibration characteristics is of interest. Thus in this case the buckled position due to the application of in-plane preloading is found by performing the static analysis. Then the small amplitude vibrations are imposed on the initially buckled plate in the form of a buckled mode. Hui (1985) and Hui and Leissa (1983) have considered this problem using a single mode analysis for homogeneous and laminated plates.

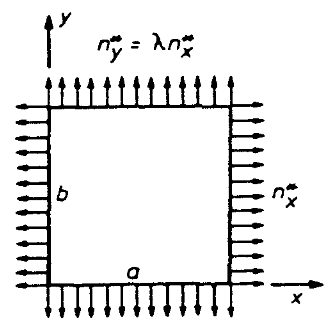

Fig. 1. Coordinate system and dimensions of the plate.

The multimode analysis using the Ritz method is given by Ilanko and Dickinson (1987a) for isotropic plates. Also, Ilanko and Dickinson (1987b) have carried out very useful experimental investigations on isotropic plates subjected to in-plane preload.

In all the above investigations the classical thin plate theory has been the basis for governing equilibrium equations. It is needless to stress the importance of inclusion of shear deformation effects in the analysis of thick and even thin orthotropic plates. Bhimaraddi (1987b, 1989) has given a single mode solution to both types of problems using the recently proposed shear deformation theory due to Bhimaraddi (1987a) but the effects of in-plane inertia and the shear rotary inertia were neglected.

It is difficult to take into account these effects when one performs a straightforward analysis such as those given by Ambartsumian *et al.* (1966), Bhimaraddi (1987b). However, as shown by Pasic and Herrmann (1984) a regular perturbation solution to the problem renders possible the inclusion of inplane and rotary inertia effects. In this paper the problem of an imperfect orthotropic plate with in-plane loading has been considered using the perturbation technique. Also, the effects of shear deformation, rotary inertia, and in-plane inertia have been taken into account.

ANALYSIS FOR IN-PLANE RESPONSE OF PLATE

The middle surface strains incorporating large deformations in the sense of von Karman are given as (refer to Fig. 1)

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w_0}{\partial y}\right)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w_0}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w_0}{\partial x}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \quad (1)$$

where u, v are the in-plane displacements in the x and y directions; w is the lateral displacement measured from the initially imperfect position (w_0) of the plate. We introduce the following dimensionless quantities for convenience.

$$\delta = h/a;\ X = x/a;\ Y = y/a;\ U = u/a;\ V = v/a;\ W = w/a;\ W_0 = w_0/\delta a;$$

$$c^2 = E_1/\rho(1-\nu_1\nu_2);\ \tau = ct\delta/\sqrt{12}a;\ \eta_2 = E_2/E_1;\ \eta_{12} = G_{12}(1-\nu_1\nu_2)/E_1;$$

$$\eta_{13} = G_{13}(1-\nu_1\nu_2)/E_1;\ \eta_{23} = G_{23}(1-\nu_1\nu_2)/E_1;\ N_x^* = n_x^*/K\delta^2;\quad N_y^* = n_y^*/K\delta^2;$$

$$\lambda = n_y^*/n_x^*;\ K = E_1h/(1-\nu_1\nu_2);\ E_2\nu_1 = \nu_2E_1;\ r = a/b. \quad (2)$$

In the above h, a, b are the thickness, length, and width of plate; t is the time coordinate; E_1, E_2 are the Young's moduli in the x and y direction; ν_1 and ν_2 are the Poisson's ratios; G_{12} is the in-plane shear modulus; G_{13}, G_{23} are the transverse shear moduli; n_x^*, n_y^* are the applied in-plane loads in the x and y direction; ρ is the mass density of the material. The in-plane membrane forces are given as (Bhimaraddi, 1987a, b, 1989)

$$\begin{aligned} N_x &= n_x/K = U' + \tfrac{1}{2}W'^2 + \delta W' W_0' + \nu_2(V^{\cdot} + \tfrac{1}{2}W_0^{\cdot 2} + \delta W^{\cdot} W_0^{\cdot}) \\ N_y &= n_y/K = \nu_2(U' + \tfrac{1}{2}W'^2 + \delta W' W_0') + \eta_2(V^{\cdot} + \tfrac{1}{2}W_0^{\cdot 2} + \delta W^{\cdot} W_0^{\cdot}) \\ N_{xy} &= n_{xy}/K = \eta_{12}(U^{\cdot} + V' + W' W^{\cdot} + \delta W' W_0^{\cdot} + \delta W^{\cdot} W_0'). \end{aligned} \tag{3}$$

Here $(\ )' = \partial/\partial X$ and $(\ )^{\cdot} = \partial/\partial Y$. In the present shear deformation theory, five generalized displacement parameters have been employed to describe the displacement components at a given point (x, y, z). They are of the form

$$\bar{u} = u + \xi\phi - z\partial w/\partial x;\ \bar{v} = v + \xi\psi - z\partial w/\partial y;\ \bar{w} = w;\ \xi = z(1 - 4z^2/3h^2) \tag{4}$$

where $\bar{u}, \bar{v}, \bar{w}$ are the displacements in the x, y, z directions at any point (x, y, z); ϕ, ψ are the shear rotations in addition to the familiar flexural rotations $\partial w/\partial x$ and $\partial w/\partial y$. Detailed discussion regarding the selection of the above displacement forms (4) and the derivation of equilibrium equations and the associated boundary conditions can be found in a paper by Bhimaraddi (1987a). Obviously there are five equilibrium equations: two corresponding to u and v; two corresponding to ϕ and ψ; and one corresponding to w displacement parameter. Equilibrium equations governing the in-plane motion are written as

$$\begin{aligned} U'' + \eta_{12}U^{\cdot\cdot} + (\nu_2 + \eta_{12})V'^{\cdot} &= -W' W'' - (\nu_2 + \eta_{12})W^{\cdot} W'^{\cdot} \\ &\quad - \eta_{12}W' W^{\cdot\cdot} - \delta(W'' W_0' + W' W_0'') - \delta\nu_2(W'^{\cdot} W_0^{\cdot} + W^{\cdot} W_0'^{\cdot}) \\ &\quad - \delta\eta_{12}(W'^{\cdot} W_0^{\cdot} + W' W_0^{\cdot\cdot} + W^{\cdot\cdot} W_0' + W^{\cdot} W_0'^{\cdot}) + \frac{\delta^2}{12}\ddot{U} \\ (\nu_2 + \eta_{12})U'^{\cdot} + \eta_2 V^{\cdot\cdot} + \eta_{12}V'' &= -\eta_2 W^{\cdot} W^{\cdot\cdot} - (\nu_2 + \eta_{12})W' W'^{\cdot} \\ &\quad - \eta_{12}W^{\cdot} W'' - \delta\eta_2(W^{\cdot\cdot} W_0^{\cdot} + W^{\cdot} W_0^{\cdot\cdot}) - \delta\nu_2(W'^{\cdot} W_0' + W' W_0'^{\cdot}) \\ &\quad - \eta_{12}\delta(W'' W_0^{\cdot} + W' W_0'^{\cdot} + W'^{\cdot} W_0' + W^{\cdot} W_0'') + \frac{\delta^2}{12}\ddot{V}. \end{aligned} \tag{5}$$

In the above, superposed dots indicate the differentiation with respect to τ. For an alround simply supported plate the boundary conditions are given as

$$\begin{aligned} &U = W = W'' = 0 \quad \text{at } X = 0; \quad W = W'' = 0 \quad \text{at } X = 1 \\ &V = W = W^{\cdot\cdot} = 0 \quad \text{at } Y = 0; \quad W = W^{\cdot\cdot} = 0 \quad \text{at } Y = 1/r \\ &r\int_0^{1/r} N_x \,\mathrm{d}Y = \delta^2 N_x^* \quad \text{at } X = 1; \quad \int_0^1 N_y \,\mathrm{d}X = \delta^2 N_y^* \quad \text{at } Y = 1/r. \end{aligned} \tag{6}$$

Since the displacements U and V are one order higher than the displacement W the following perturbation series due to Pasic and Herrmann (1984) are used.

$$U = \delta^2 U_1 + \delta^4 U_2 + \cdots; \quad V = \delta^2 V_1 + \delta^4 V_2 + \cdots; \quad W = \delta W_1 + \cdots \tag{7}$$

Substituting the series (7) into eqns (5) and equating the like powers of δ one obtains the following system of equations

$$\begin{aligned} U_1'' + \eta_{12}U_1^{\cdot\cdot} + (\nu_2 + \eta_{12})V_1'^{\cdot} &= -W_1' W_1'' - (\nu_2 + \eta_{12})W_1^{\cdot} W_1'^{\cdot} \\ &\quad - \eta_{12}W_1' W_1^{\cdot\cdot} - (W_1'' W_0' + W_1' W_0'') - \nu_2(W_1'^{\cdot} W_0^{\cdot} + W_1^{\cdot} W_0'^{\cdot}) \\ &\quad - \eta_{12}(W_1'^{\cdot} W_0^{\cdot} + W_1' W_0^{\cdot\cdot} + W_1^{\cdot\cdot} W_0' + W_1^{\cdot} W_0'^{\cdot}) \end{aligned}$$

$$(\nu_2+\eta_{12})U_1^{\cdot\prime}+\eta_2 V_1^{\cdot\cdot}+\eta_{12}V_1'' = -\eta_2 W_1^{\cdot}W_1^{\cdot\cdot}-(\nu_2+\eta_{12})W_1'W_1^{\prime\cdot}-\eta_{12}W_1^{\cdot}W_1''$$
$$-\eta_2(W_1^{\cdot\cdot}W_0^{\cdot}+W_1^{\cdot}W_0^{\cdot\cdot})-\nu_2(W_1^{\cdot\prime}W_0'+W_1'W_0^{\cdot\prime})$$
$$-\eta_{12}(W_1''W_0^{\cdot}+W_1'W_0^{\cdot\prime}+W_1^{\cdot\prime}W_0'+W_1^{\cdot}W_0'') \tag{8}$$

$$U_2''+\eta_{12}U_2^{\cdot\cdot}+(\nu_2+\eta_{12})V_2^{\cdot\prime} = \tfrac{1}{12}\ddot{U}_1$$
$$(\nu_2+\eta_{12})U_2^{\cdot\prime}+\eta_2 V_2^{\cdot\cdot}+\eta_{12}V_2'' = \tfrac{1}{12}\ddot{V}_1. \tag{9}$$

The boundary conditions (6) become

$$U_1 = W_1 = W_1'' = U_2 = 0 \quad \text{at } X=0; \quad W_1 = W_1'' = 0 \quad \text{at } X=1$$
$$V_1 = W_1 = W_1^{\cdot\cdot} = V_2 = 0 \quad \text{at } Y=0; \quad W_1 = W_1^{\cdot\cdot} = 0 \quad \text{at } Y=1/r \tag{10}$$

$$r\int_0^{1/r} N_{x1}\,\mathrm{d}Y = N_x^*; \quad \int_0^{1/r} N_{x2}\,\mathrm{d}Y = 0 \quad \text{at } X=1$$
$$\int_0^1 N_{y1}\,\mathrm{d}X = N_y^*; \quad \int_0^1 N_{y2}\,\mathrm{d}X = 0 \quad \text{at } Y=1/r \tag{11}$$

in which N_{x1}, N_{x2} etc. are given by

$$N_{x1} = U_1'+\tfrac{1}{2}W_1'^2+W_1'W_0'+\nu_2(V_1^{\cdot}+\tfrac{1}{2}W_0^{\cdot 2}+W_1^{\cdot}W_0^{\cdot})$$
$$N_{y1} = \nu_2(U_1'+\tfrac{1}{2}W_1'^2+W_1'W_0')+\eta_2(V_1^{\cdot}+\tfrac{1}{2}W_0^{\cdot 2}+W_1^{\cdot}W_0^{\cdot})$$
$$N_{xy1} = \eta_{12}(U_1^{\cdot}+V_1'+W_1'W_1^{\cdot}+W_1^{\cdot}W_0'+W_1'W_0^{\cdot}) \tag{12}$$

$$N_{x2} = U_2'+\nu_2 V_2^{\cdot}; \quad N_{y2} = \nu_2 U_2'+\eta_2 V_2^{\cdot}; \quad N_{xy2} = \eta_{12}(U_2^{\cdot}+V_2'). \tag{13}$$

The simply supported boundary conditions are satisfied by using the transverse displacement of the form

$$W_1 = f(\tau)\sin\pi X\sin\pi rY \tag{14}$$

and the initial imperfections are assumed to be of the type

$$W_0 = f_0\sin\pi X\sin\pi rY. \tag{15}$$

Substituting eqns (14) and (15) into (8) and using the boundary conditions (10) and (11) we obtain the following expressions for U_1 and V_1.

$$U_1 = (f^2+2ff_0)(a_1\sin 2\pi X+a_2\sin 2\pi X\cos 2\pi rY+a_3X)+(b_1N_x^*+b_2N_y^*)X$$
$$V_1 = (f^2+2ff_0)(a_4\sin 2\pi rY+a_5\cos 2\pi X\sin 2\pi rY+a_6Y)+(b_2N_x^*+b_3N_y^*)Y. \tag{16}$$

Substituting eqns (14)–(16) into eqns (9) and using the boundary conditions (10) and (11), pertaining to U_2 and V_2 displacements, we obtain the following expressions for U_2 and V_2.

$$U_2 = 2(f\ddot{f}+\dot{f}^2+f_0\ddot{f})\left(a_9\sin 2\pi X+a_{10}\sin 2\pi X\cos 2\pi rY+a_{11}X+a_3\frac{X^3}{72}\right)$$
$$+(b_1\ddot{N}_x^*+b_2\ddot{N}_y^*)\left(\frac{X^3}{72}+b_4X\right)+(b_2\ddot{N}_x^*+b_3\ddot{N}_y^*)b_5X$$

$$V_2 = 2(f\ddot{f}+\dot{f}^2+f_0\ddot{f})\left(a_{12}\sin 2\pi rY+a_{13}\cos 2\pi X\sin 2\pi rY+a_{14}Y+a_6\frac{Y^3}{72}\right)$$
$$+(b_1\ddot{N}_x^*+b_2\ddot{N}_y^*)b_6Y+(b_2\ddot{N}_x^*+b_3\ddot{N}_y^*)\left(b_7Y+\frac{Y^3}{72}\right). \tag{17}$$

Using the solutions (16) and (17) in eqns (12) and (13) the following expressions for in-plane forces can be obtained.

$$N_{x1} = (f^2+2ff_0)a_7\cos 2\pi rY+N_x^*$$
$$N_{y1} = (f^2+2ff_0)a_8\cos 2\pi X+N_y^*;\quad N_{xy1}=0 \tag{18}$$

$$N_{x2} = 2(f\ddot{f}+\dot{f}^2+f_0\ddot{f})(a_{15}\cos 2\pi X+a_{16}\cos 2\pi rY+a_{17}\cos 2\pi X\cos 2\pi rY$$
$$+a_{18}+a_{19}X^2+a_{20}Y^2)+(b_1\ddot{N}_x^*+b_2\ddot{N}_y^*)\left(b_8+\frac{X^2}{24}\right)+(b_2\ddot{N}_x^*+b_3\ddot{N}_y^*)\left(b_9+v_2\frac{Y^2}{24}\right)$$
$$N_{y2} = 2(f\ddot{f}+\dot{f}^2+f_0\ddot{f})(a_{21}\cos 2\pi X+a_{22}\cos 2\pi rY+a_{23}\cos 2\pi X\cos 2\pi rY+a_{24}$$
$$+a_{25}X^2+a_{26}Y^2)+\left(b_1\ddot{N}_x^*+b_2\ddot{N}_y^*\right)\left(b_{10}+v_2\frac{X^2}{24}\right)+(b_2\ddot{N}_x^*+b_3\ddot{N}_y^*)\left(b_{11}+\eta_2\frac{Y^2}{24}\right)$$
$$N_{xy2} = 2(f\ddot{f}+\dot{f}^2+f_0\ddot{f})a_{27}\sin 2\pi X\sin 2\pi rY. \tag{19}$$

This completes the analysis for in-plane response in which the lateral displacement (14) is the only unknown to be determined. In the next section we consider the inclusion of transverse shear and rotary inertia effects.

INCLUSION OF TRANSVERSE SHEAR AND ROTARY INERTIA EFFECTS

The two equilibrium equations governing the transverse shear response are written in dimensionless form as

$$\phi''+\eta_{12}\phi^{\cdot\cdot}-\frac{168}{17\delta^2}\eta_{13}\phi+(v_2+\eta_{12})\psi^{\cdot\prime} = \frac{21}{17}W'''+\frac{21}{17}(v_2+2\eta_{12})W'^{\cdot\cdot}+\frac{\delta^2}{12}\ddot{\phi}-\frac{7\delta^2}{68}\ddot{W}'$$
$$\eta_2\psi^{\cdot\cdot}+\eta_{12}\psi''-\frac{168}{17\delta^2}\eta_{23}\psi+(v_2+\eta_{12})\phi^{\cdot\prime} = \frac{21}{17}\eta_2W^{\cdot\cdot\cdot}+\frac{21}{17}(v_2+2\eta_{12})W''^{\cdot}+\frac{\delta^2}{12}\ddot{\psi}-\frac{7\delta^2}{68}\ddot{W}^{\cdot}. \tag{20}$$

It is well-known that the shear rotations are one order higher than the in-plane displacements. Thus, we select the following series for two shear rotations ϕ and ψ:

$$\phi = \delta^3\phi_1+\delta^5\phi_2+\delta^7\phi_3+\cdots;\quad \psi = \delta^3\psi_1+\delta^5\psi_2+\delta^7\psi_3+\cdots \tag{21}$$

It may further be noted that any other choice, than series (21), would have been inconsistent with eqns (20) as there are terms in which δ appears in the denominator. Substituting the series (21) and (14) into eqns (20) we obtain the following set of equations in terms of ϕ_is and ψ_is.

$$8\eta_{13}\phi_1 = \pi^3(1+r^2v_2+2r^2\eta_{12})f\cos\pi X\sin\pi rY$$
$$8\eta_{23}\psi_1 = \pi^3r(v_2+r^2\eta_2+2\eta_{12})f\sin\pi X\cos\pi rY \tag{22}$$

$$\frac{168}{17}\eta_{13}\phi_2 = \phi_1'' + \eta_{12}\phi_1^{\cdot\cdot} + (v_2+\eta_{12})\psi_1^{\cdot\prime} + \frac{7\pi}{68}\ddot{f}\cos\pi X\sin\pi rY$$

$$\frac{168}{17}\eta_{23}\psi_2 = \eta_2\psi_1^{\cdot\cdot} + \eta_{12}\psi_1'' + (v_2+\eta_{12})\phi_1^{\cdot\prime} + \frac{7r\pi}{68}\ddot{f}\sin\pi X\cos\pi rY \tag{23}$$

$$\frac{168}{17}\eta_{13}\phi_3 = \phi_2'' + \eta_{12}\phi_2^{\cdot\cdot} + (v_2+\eta_{12})\psi_2^{\cdot\prime} - \frac{1}{12}\ddot{\phi}_1$$

$$\frac{168}{17}\eta_{23}\psi_3 = \eta_2\psi_2^{\cdot\cdot} + \eta_{12}\psi_2'' + (v_2+\eta_{12})\phi_2^{\cdot\prime} - \frac{1}{12}\ddot{\psi}_1. \tag{24}$$

Solving the above equations successively we obtain the following expressions for ϕ_is and ψ_is.

$$\phi_1 = a_{28}f\cos\pi X\sin\pi rY; \qquad \psi_1 = a_{29}f\sin\pi X\cos\pi rY$$

$$\phi_2 = (a_{30}f + a_{31}\ddot{f})\cos\pi X\sin\pi rY; \quad \psi_2 = (a_{32}f + a_{33}\ddot{f})\sin\pi X\cos\pi rY$$

$$\phi_3 = (a_{34}f + a_{35}\ddot{f})\cos\pi X\sin\pi rY; \quad \psi_3 = (a_{36}f + a_{37}\ddot{f})\sin\pi X\cos\pi rY. \tag{25}$$

Again, we note that the only unknown to be determined in eqns (25) is the transverse displacement which can be determined by analysing the lateral motion.

ANALYSIS OF OUT-OF-PLANE RESPONSE OF PLATE

The equation governing the lateral motion of the plate may be written in dimensionless form as (Bhimaraddi, 1987b)

$$\begin{aligned}
&-\frac{12}{15}\phi''' - \frac{12}{15}(v_2+2\eta_{12})(\phi^{\cdot\cdot\prime} + \psi^{\cdot\prime\prime}) - \frac{12}{15}\eta_2\psi^{\cdot\cdot\cdot} + W^{\mathrm{IV}} \\
&+ 2(v_2+2\eta_{12})W^{\cdot\cdot\prime\prime} + \eta_2 W^{::} = 12(N_{x1}+\delta^2 N_{x2})(W''+\delta W_0'') \\
&+ 12(N_{y1}+\delta^2 N_{y2})(W^{\cdot\cdot}+\delta W_0^{\cdot\cdot}) + 24\delta^2 N_{xy2}(W^{\cdot\prime}+\delta W_0^{\cdot\prime}) \\
&+ 12(N_{x1}'+\delta^2 N_{x2}'+\delta^2 N_{xy2}^{\cdot})(W'+\delta W_0') \\
&+ 12(N_{y1}^{\cdot}+\delta^2 N_{y2}^{\cdot}+\delta^2 N_{xy2}')(W^{\cdot}+\delta W_0^{\cdot}) \\
&+ \frac{12a}{K\delta^2}q - \ddot{W} + \frac{\delta^2}{12}(\ddot{W}'' + \ddot{W}^{\cdot\cdot}) + \frac{\delta^2}{15}(\ddot{\phi}' + \ddot{\psi}^{\cdot})
\end{aligned} \tag{26}$$

where q is the applied load in the lateral direction on the plate surface. All the quantities in the above equation have already been expressed in terms of a single quantity $f(\tau)$. Substituting the same and applying the Galerkin's method the following equation is obtained.

$$\begin{aligned}
&(1+\delta^2 a_{38}+\delta^4 a_{39}+\delta^6 a_{40}+\delta^8 a_{41})\ddot{f} + (a_{42}+\delta^2 a_{43}+\delta^4 a_{44}+\delta^6 a_{45})f \\
&\quad + [12\pi^2(N_x^*+r^2N_y^*)+\delta^2(\ddot{N}_x^* a_{46}+\ddot{N}_y^* a_{47})](f+f_0) + a_{48}(f^2+2ff_0)(f+f_0) \\
&\qquad + \delta^2 a_{49}(f\ddot{f}+\dot{f}^2+\ddot{f}f_0)(f+f_0) + (\delta^6 a_{50}+\delta^8 a_{51})\overset{::}{f} = Q.
\end{aligned} \tag{27}$$

Some comments regarding the various terms in the above equation are in order. It may be seen that a_{38} is the contribution due to the inclusion of rotary inertia of the flexural rotations and a_{39} to a_{41} are the contributions due to the inclusion of rotary inertia of the shear rotations to the total inertia. The terms from a_{43} to a_{45} are the contributions due to the inclusion shear deformation to the stiffness term. The terms containing a_{46}, a_{47} and a_{49} are due to the inclusion of in-plane inertia. Also the terms containing a_{50} and a_{51} arise due to the inclusion of shear rotary inertia and they are neglected in the subsequent treatment due to their smallness as compared to other terms.

It is to be noted here that the inclusion of the effect of rotary inertia is to increase the inertia and that of shear deformation is to decrease the stiffness of the system, as compared to the one when they are neglected. Further, we note that the correction to the inertia term due to the inclusion of rotary inertia of the shear rotations is very small as compared to that of the flexural rotations and hence one can safely neglect the same. By using $F = f + f_0$ eqn (27) can be written as

$$\alpha_1 \ddot{F} + \alpha_2(1 + N + \alpha_3 \ddot{N} - \alpha_4 f_0^2/\alpha_2)F + \alpha_4 F^3 + \alpha_5(F^2\ddot{F} + F\dot{F}^2) = \alpha_2 f_0 + Q \tag{28}$$

and the α's in the above equations are defined as

$$\alpha_1 = 1 + \delta^2 a_{38} + \delta^4 a_{39} + \delta^6 a_{40} + \delta^8 a_{41}; \quad \alpha_2 = a_{42} + \delta^2 a_{43} + \delta^4 a_{44} + \delta^6 a_{45}$$
$$\alpha_3 = \delta^2(a_{46} + \lambda a_{47})/12\pi^2(1 + \lambda r^2); \quad \alpha_4 = a_{48}; \quad \alpha_5 = \delta^2 a_{49}$$
$$\lambda = N_y^*/N_x^*; \quad N = N_x^*/N_c; \quad N_c = \alpha_2/12\pi^2(1 + \lambda r^2); \quad Q = 192aq/K\delta^3\pi^2. \tag{29}$$

Here we have assumed proportional in-plane loading in the x and y directions and N_c corresponds to the static bifurcation buckling load of biaxially compressed perfect plate. Equation (28) is the most general equation governing the nonlinear dynamic response of an orthotropic plate subjected to in-plane and lateral loading, incorporating the effects of rotary inertia and shear deformation. This equation is exactly of the same form as that derived by Pasic and Herrmann (1984) for isotropic plates. In this paper, among others, we consider the large amplitude oscillations, parametric excitations, and forced harmonic oscillations of orthotropic plates. The following orthotropic material properties are used in the numerical examples:

$$E_1/E_2 = 20; \quad E_1/G_{12} = E_1/G_{13} = E_1/G_{23} = 40; \quad \nu_1 = 0.25$$

and the Poisson's ratio in the case of an isotropic plate examples is taken as 0.3.

LARGE AMPLITUDE VIBRATIONS OF AN IN-PLANE LOADED PLATE

Under the absence of lateral loading ($q = 0$) and time independent in-plane loading ($\ddot{N} = 0$) eqn (28) can be transformed as

$$\ddot{F} + F + \varepsilon F^3 + \gamma(F^2\ddot{F} + F\dot{F}^2) = \mu. \tag{30}$$

Here superposed dots indicate differentiation with respect to $(\Omega_L \tau)$, and we have used the following definitions.

$$\Omega_L = \sqrt{(\alpha_2(1 + N - \alpha_4 f_0^2/\alpha_2)/\alpha_1)}; \quad \varepsilon = \alpha_4/(1 + N - \alpha_4 f_0^2/\alpha_2)\alpha_2$$
$$\gamma = \alpha_5/\alpha_1; \quad \mu = f_0/(1 + N - \alpha_4 f_0^2/\alpha_2). \tag{31}$$

using the method of multiple scales (Nayfeh and Mook, 1979) the solution of eqn (30) for perfect plates ($f_0 = 0$) can be written as

$$F = A\cos\theta + c_1 A^3 \cos 3\theta + 0(\varepsilon^3) \tag{32}$$

where $\theta = \Omega_{NL}\tau + \beta$; A and β are the constants to be determined from the initial conditions

Table 1. Linear frequencies (Ω_L) of square isotropic plates ($\delta = 0.1$, $\lambda = 1$)

f_0	Classical theory NRI	Classical theory FRI	Shear deformation theory NRI	Shear deformation theory FRI	Shear deformation theory SRI
			$N = 0$		
0.0	19.739	19.579	19.210	19.054	19.060
0.1	19.807	19.646	19.279	19.123	19.128
0.2	20.007	19.844	19.485	19.327	19.333
			$N = -0.4$		
0.0	15.557	15.431	14.880	14.759	14.764
0.1	15.642	15.515	14.969	14.848	14.852
0.2	15.895	15.766	15.233	15.109	15.114

NRI, rotary inertias neglected; FRI, flexural rotary inertia included; SRI, both flexural and shear rotary inertias included.

and Ω_{NL} is the nonlinear frequency which is dependent on the amplitude A in the following manner:

$$\Omega_{NL} = [1+(3\varepsilon-2\gamma)A^2/8]\Omega_L. \tag{33}$$

It is clear from the above expression that the effect of geometric nonlinearity is to increase the frequency, whereas the effect of nonlinear inertia is to decrease the same. The same conclusion has been made by Pasic and Herrmann (1984) in their study of rectangular plates and by Bolotin (1964) in the study of beams. The constant c_1 appearing in eqn (32) has the following definition

$$c_1 = (\varepsilon-2\gamma)/32. \tag{34}$$

It may be observed from Table 1 and 2 that the shear deformation effects on linear frequency become increasingly dominant with increasing in-plane load and the initial imperfections. Even in the case of isotropic plate the difference between CPT and SDT is about 5% ($N = -0.4$, $f_0 = 0.2$). As noted earlier, the influence of shear rotary inertia on the frequency is very small and can be neglected altogether. The material anisotropy requires the use of shear deformation theory as there is a difference of more than 20% between the results of the classical (CPT) and the shear deformation (SDT) theories.

Tables 3 and 4 depict the nonlinear frequencies for in-plane loaded perfect square plates. We note that the influence of in-plane inertia is insignificant even as the amplitude of vibration increases. There is about 0.25% difference between the frequencies with in-plane inertia and without in-plane inertia being considered. This difference does not seem to depend much on the magnitudes of in-plane load.

Table 2. Linear frequencies (Ω_L) of square orthotropic plates ($\delta = 0.1$, $\lambda = 1$)

f_0	Classical theory NRI	Classical theory FRI	Shear deformation theory NRI	Shear deformation theory FRI	Shear deformation theory SRI
			$N = 0$		
0.0	10.697	10.610	9.015	8.942	8.953
0.1	10.832	10.744	9.176	9.101	9.112
0.2	11.228	11.137	9.640	9.562	9.573
			$N = -0.4$		
0.0	9.051	8.977	6.983	6.927	6.935
0.1	9.210	9.135	7.189	7.130	7.139
0.2	9.673	9.594	7.773	7.710	7.719

Table 3. Nonlinear to linear frequency ratios (Ω_{NL}/Ω_L)† for in-plane loaded perfect square isotropic plates ($\delta = 0.1$, $\lambda = 1$)

	Without in-plane inertia			With in-plane inertia		
	$A = 1/5$	$A = 3/5$	$A = 1$	$A = 1/5$	$A = 3/5$	$A = 1$
			$N = 0$			
CPT	1.033	1.075	1.159	1.032	1.073	1.155
SDT	1.005	1.050	1.135	1.005	1.047	1.132
			$N = -0.4$			
CPT	1.053	1.123	1.261	1.054	1.121	1.257
SDT	1.009	1.081	1.225	1.009	1.080	1.222

† Linear frequency corresponds to the value of SDT.
CPT, classical plate theory; SDT, shear deformation theory.

Table 4. Nonlinear to linear frequency ratios (Ω_{NL}/Ω_L) for in-plane loaded perfect square orthotropic plates ($\delta = 0.1$, $\lambda = 1$)

	Without in-plane inertia			With in-plane inertia		
	$A = 1/5$	$A = 3/5$	$A = 1$	$A = 1/5$	$A = 3/5$	$A = 1$
			$N = 0$			
CPT	1.208	1.389	1.751	1.208	1.394	1.764
SDT	1.027	1.246	1.672	1.027	1.246	1.683
			$N = -0.4$			
CPT	1.329	1.605	2.158	1.330	1.611	2.172
SDT	1.045	1.403	2.120	1.045	1.407	2.131

Table 5. Steady-state amplitude values of parametrically excited square orthotropic perfect plate ($P = -0.2$; $\lambda = 1$; SDT results)

$\delta = 0.1$	$\delta = 0.05$	$\delta = 0.025$	$\delta = 0.01$
0.39461†	0.44063	0.45323	0.45697
0.38575‡	0.43730	0.45231	0.45682

† Including in-plane inertia.
‡ Neglecting in-plane inertia.

PRINCIPAL PARAMETRIC OSCILLATIONS

In this section we consider the case of the subharmonic resonance of order two for perfect plates ($f_0 = 0$). In this case eqn (28), under the absence of lateral load ($q = 0$) and under the presence of time dependent inplane load ($N = P\cos 2\tau'$), can be written as

$$\ddot{F}+(1+\Delta\cos 2\tau')F+\varepsilon F^3+\gamma(F^2\ddot{F}+F\dot{F}^2) = 0. \tag{35}$$

Here the superposed dots indicate the differentiation with respect to τ', and we have used the following definitions for coefficients appearing in the above equation.

$$\tau' = \sqrt{(\alpha_2/\alpha_1\tau)};\quad \Delta = P(\alpha_1-4\alpha_3\alpha_2)/\alpha_1;\quad \varepsilon = \alpha_4/\alpha_2;\quad \gamma = \alpha_5/\alpha_1. \tag{36}$$

Again using the method of multiple scales and restricting ourselves to the first approximation, the solution of eqn (35) can be written as

$$F = A\cos\tau'+0(\varepsilon). \tag{37}$$

Here A is the steady-state amplitude which is given by the following relation

$$A = \pm\sqrt{(-4\Delta/(3\varepsilon-2\gamma))}. \tag{38}$$

Table 5 shows steady-state amplitude [eqn (38)] values for orthotropic square plate with various thickness to length ratios. It is clearly observable that the effects of in-plane

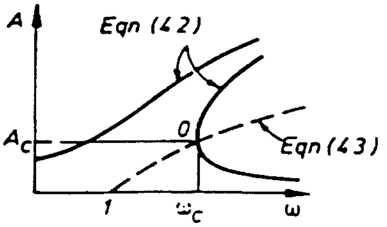

Fig. 2. Nonlinear response curve.

inertia are predominant in thick plates and their effects become less predominant as the thickness of the plate decreases. For 10% thick plate the difference in the amplitudes is about 2.3% between the results when the in-plane inertia is considered and is not considered. Whereas, for 1% thick plate this difference is only 0.04%.

FORCED RESPONSE OF PLATES SUBJECTED TO HARMONIC LATERAL PRESSURE

In this section we consider the nonlinear response of plates subjected to harmonically varying lateral pressure ($q = p\cos\omega\tau'$) on the surface. Equation (28) for perfect plates, under the absence of in-plane loads, can be written as

$$\ddot{F}+F+\varepsilon F^3+\gamma(F^2\ddot{F}+F\dot{F}^2) = k\cos\omega\tau'. \tag{39}$$

Here superposed dots indicate differentiation with respect to τ' and ω is the frequency of the applied pressure. We have used the following definitions.

$$\tau' = \sqrt{(\alpha_2/\alpha_1\tau)}; \quad k = 192pa/K\pi^2\delta^3\alpha_2; \quad \varepsilon = \alpha_4/\alpha_2; \quad \gamma = \alpha_5/\alpha_1. \tag{40}$$

The solution of eqn (39) to first approximation can be written as

$$F = A\cos\omega\tau' \tag{41}$$

where A is the steady-state amplitude. The relation between A and the applied load parameters (k, ω) can be obtained using the method of multiple scales as

$$\omega = A^2(3\varepsilon-2\gamma)/8 - k/2A + 1. \tag{42}$$

The typical plot of eqn (42) is shown in Fig. 2. Note that point "0" in Fig. 2 corresponds to the point where the motion becomes unstable or the "jump" phenomenon occurs. Also at this point we have $\mathrm{d}\omega/\mathrm{d}A = 0$. Utilising this fact and eqn (42) we obtain the following relation.

$$\omega = 3A^2(3\varepsilon-2\gamma)/8. \tag{43}$$

The plot of this equation is also shown in Fig. 2. Thus the critical amplitude (A_c) and critical forcing frequency (ω_c), which are the coordinates of the point "0", can be obtained using eqns (42) and (43) as

$$A_c = -\left[\frac{2k}{(3\varepsilon-2\gamma)}\right]^{1/3}; \quad \omega_c = 1+\left[\frac{27k^2}{128}(3\varepsilon-2\gamma)\right]^{1/3}. \tag{44}$$

Some numerical results depicting the values of the critical frequency are shown in

Table 6. Critical forcing frequency (ω_c) values of orthotropic square plate (SDT results)

k	$\delta = 0.1$		$\delta = 0.05$	
1	2.0428†	2.0487‡	1.9591†	1.9609‡
2	2.6554	2.6647	2.5225	2.5254
3	3.1692	3.1814	2.9951	2.9988

† Neglecting in-plane inertia.
‡ Including in-plane inertia.

Table 6 for different values of plate thickness and magnitude of the applied load. It is evident from this table that the effects of in-plane inertia is insignificant even in the case of 10% thick plate.

NONLINEAR VIBRATIONS OF PRELOADED (IN-PLANE) PLATES

Here we consider the second kind of in-plane loaded plate vibration problem and study the influence of in-plane inertia on the vibration characteristics of the preloaded plate about its static deflected position. The governing equation corresponding to the static response, under the absence of laterally applied load ($q = 0$) and after ignoring the time derivatives, can be written from eqn (28) as

$$\alpha_2(1+N-\alpha_4^2 f_0/\alpha_2)F_s+\alpha_4 F_s^3 = \alpha_2 f_0. \tag{45}$$

Here F_s represents (refer to Fig. 3) the deflection of the in-plane loaded imperfect plate. If the plate is imperfection sensitive then the limit load $N_L < 1$. Now we superimpose the large amplitude vibrations (f_d), in the form of the deflected shape, over the static deflected position. Substituting f_d+F_s in place of F in eqn (28) and noting that the time derivatives of F_s vanish, we obtain the following equation.

$$\ddot{f}_d+\omega_s^2 f_d+\alpha f_d^2+\varepsilon f_d^3+\beta(\dot{f}_d^2+2f_d\ddot{f}_d)+\gamma(f_d^2\ddot{f}_d+f_d\dot{f}_d^2) = 0 \tag{46}$$

where ω_s represents linear frequency of the in-plane preloaded plate and the following definitions have been used.

$$\omega_s^2 = \frac{\alpha_2(1+N)-\alpha_4 f_0^2+3\alpha_4 F_s^2}{\alpha_5 F_s^2+\alpha_1}; \quad \alpha = \frac{3\alpha_4 F_s}{\alpha_5 F_s^2+\alpha_1}$$

$$\varepsilon = \frac{\alpha_4}{\alpha_5 F_s^2+\alpha_1}; \quad \beta = \frac{\alpha_5 F_s}{\alpha_5 F_s^2+\alpha_1}; \quad \gamma = \frac{\alpha_5}{\alpha_5 F_s^2+\alpha_1}. \tag{47}$$

It is evident from the presence of α_5 term in the frequency expression that the frequency of the preloaded plate is also effected by in-plane inertia. However, in this case when the plate is vibrating about its static deflected position the in-plane inertia increases the frequency value unlike the case of a plate vibrating about its initial unloaded position where the effect of in-plane inertia is to decrease the same. It can easily be shown that the frequency

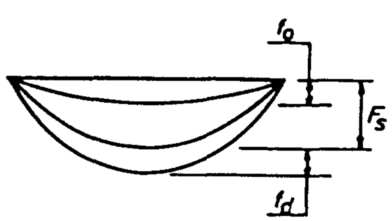

Fig. 3. Deflected positions of in-plane loaded plate.

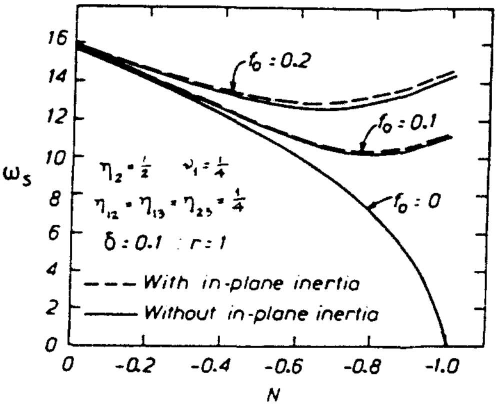

Fig. 4. Variation of frequency of an orthotropic plate with in-plane preloading.

of a preloaded plate vanishes at the limit load. At limit load we have $dN/dF_s = 0$. Using this fact and eqn (45) one can obtain the following equation:

$$2\alpha_4 F_L^3 - \alpha_2 F_L + \alpha_2(1+f_0) = 0. \tag{48}$$

Substituting F_L in place of F_s one obtains an expression for N_L from eqn (45) in terms of F_L. Then substituting this expression for N_L and also F_L in place of F in the expression for frequency, i.e. first of the eqns (47), one can see that the frequency expression reduces to that of eqn (48). The solution of eqn (46) can be obtained by the method of multiple scales which yields the expression for nonlinear frequency as

$$\omega_{NL}/\omega_s = 1 + \left[\frac{3\varepsilon}{8\omega_s^2} - \frac{5\alpha^2}{12\omega_s^4} + \frac{7}{4}\beta^2 - \frac{1}{4}\gamma\right]A_d^2. \tag{49}$$

In the above, A_d represents the amplitude of vibration of the preloaded plate about its static deflected position. Figure 4 depicts comparison of frequencies (ω_s) of a preloaded plate considering and not considering the in-plane inertia effects. It may be observed that the influence of in-plane inertia is very small and can be ignored even in the present case of 10% thick plate.

CONCLUSIONS

In conclusion we note that the regular perturbation technique to the nonlinear plate vibration analysis has been used to study the effects of shear deformation, rotary inertia, and the in-plane inertia. From the numerical results obtained for different plate vibration problems it is shown that the effects of in-plane inertia are to be considered while analysing the nonlinear response of parametrically excited plates and they can be safely neglected while analysing other types of plate vibration problems.

Acknowledgements—The author thanks Mrs Priyadarshini and Mrs D. Forbes for their help in typing the manuscript.

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